Induction Template, COMP 283

2023

The most important thing to note is that x is not always just a single element or a number. x can be a set. We will do an example of both cases.

S1. State the 'for all' statement that you want to prove: "Here, $\forall x \in S, R(x)$." Sometimes we strengthen the statement of what we want to prove in order to have stronger assumptions in S5.

S2. Say "we prove this by induction on" and state the induction parameter.
If x is a number. Typically, you will just be inducting on x as it gets bigger.
If x is a set. Typically, you will induct on the *size* of x. So often you define $n = |x|$.
If x is a function/mapping. This is a little bit trickier. Sometimes, you will induct on certain inputs/components of x. For example, if x is defined as $x = f(y, z)$, you might want to induct on just y or just z. A good example of what I mean by this is if you think way we defined addition recursively. $M(y, z)$ works by fixing z and adding 1 y times. So here, you might want to induct on y.

S3. Prove the base case(s). Here you use the definition to check the truth of $R(x)$ for specific instances of x with small values of the induction parameter. If you have trouble getting an induction rolling, do an extra base case or two.
If x is a number. Often you demonstrate it for $x = 0$ or $x = 1$ or both.
If x is a set. Often you demonstrate it for $n = 0$ or $n = 1$ or both.
If x is a function/mapping. If x is defined as $x = f(y, z)$ and you are inducting on y, often you demonstrate it for $f(0, z)$ or $f(1, z)$ or both.

S4. Write Induction Step. This mindless step is a reminder that in the induction step you are proving $R(x)$ for a specific, given instance x. No $\forall x \in S$ quantifier allowed here!

If x is a number. "For a given $x > \langle\text{base case}\rangle$, "
If x is a set. "For a given x of size $n > \langle\text{base case}\rangle$, "
If x is a function/mapping. "For a given $y > \langle\text{base case}\rangle$, "

S5. State the Induction Hypothesis (IH): This repeats the phrasing of S1, but now for all y of size $k < n$ in place of for all x, because while trying to prove $R(x)$, we get to assume that we know $R(y)$ for all ys smaller than x, including the base cases.

If x is a number. "I can assume, for all k, with $\langle\text{base cases}\rangle \leq k < x$, that. . . " (e.g., that $R(k)$ is true.)
If x is a set. "I can assume, for all z of size k, with $\langle\text{base cases}\rangle \leq k < n$, that. . . " (e.g., that $R(z)$ is true.)
If x is a function/mapping. "For x = f(y,z), I can assume, for all k, z, with $\langle\text{base cases}\rangle \leq k < y$, that. . . " (e.g., that $R(f(k, z))$ is true.)

S6. State what you are going to prove about your specific value of x that was given to you in S4: e.g., "I want to prove $R(x)$." Again, no $\forall$ quantifier, because we have a specific x of size n to work with. Sometimes S4–S6 are combined for short proofs.

S7. Do the proof for the specific x, often by expanding the basic definition, applying the IH, then doing some calculation. Once you've chosen what you plan to prove in S1, you don't really have to think until somewhere in the middle of this step.

S8. Declare victory. "Therefore, we have proved $\forall x, R(x)$ by induction."

Example 1

Recall the Fibonacci series...

1. The base case: $F(0) = 0, F(1) = 1$
2. Recursive rule: For $n > 1$, $F(n) = F(n-1) + F(n-2)$.

We want to prove the following about the sum of the first n numbers of the series.
So, we want to show: $\forall n \in \mathbb{N}, F(0) + F(1) + F(2) + \ldots + F(n-1) + F(n) = F(n+2) - 1$.
Note that our $x = F(n)$.

S1. $\forall n \in \mathbb{N}, \sum_{i=0}^{n} F(i) = F(n+2) - 1$.

S2. We prove this by induction on n.

S3. Base cases:

- $n = 0$:
  - $F(0) = F(2) - 1$
  - $0 = 0$ $\square$
- $n = 1$:
  - $F(1) = F(3) - 1$
  - $1 = 2 - 1$ $\square$

S4. For a given $n > 1$,

S5. I can assume for all $1 \leq k \leq n$ that $\sum_{i=0}^{k} F(i) = F(k+2) - 1$,

S6. and I want to prove $\sum_{i=0}^{n} F(i) = F(n+2) - 1$.

S7.

1.	$\forall 1 \leq k \leq n, \sum_{i=0}^{k} F(i) = F(k+2) - 1$	IH
2.	Let $k = n - 1$, then $\sum_{i=0}^{n-1} F(i) = F(n - 1 + 2) - 1$	Applied IH
3.	$\sum_{i=0}^{n-1} F(i) = F(n+1) - 1$	Rewrote Line 2
4.	$\sum_{i=0}^{n-1} F(i) + F(n) = F(n+1) - 1 + F(n)$	Added $F(n)$ to both sides.
5.	$F(n+2) = F(n+1) + F(n)$	Fibonacci Def.
6.	$\sum_{i=0}^{n-1} F(i) + F(n) = F(n+2) - 1$	Plugged 5. into 4.
7.	$\sum_{i=0}^{n} F(i) = \sum_{i=0}^{n-1} F(i) + F(n)$	Def. of Sum
8.	$\sum_{i=0}^{n} F(i) = F(n+2) - 1$	Plugged 7. into 6.$\square$

S8. Therefore, we have proved $\forall n \in \mathbb{N}, F(0)+F(1)+F(2)+\ldots+F(n-1)+F(n) = F(n+2)-1$.

1 Acknowledgements

Content for these lecture notes was taken from lecture notes by Jack Snoeyink (UNC) [Sno21].

References

[Sno21] Jack Snoeyink. Discrete mathematics lecture notes. 2021.